

# Anomalous Pion Decay Revisited

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## Abstract

An implicit four dimensional regularization is applied to calculate the axial-vector-vector anomalous amplitude. The present technique always complies with results of Dimensional Regularization and can be easily applied to processes involving odd numbers of  $\gamma_5$  matrices. This is illustrated explicitly in the example of this letter.

Anomalies are undoubtedly one of the most important and intriguing aspects of Quantum Field Theory. Their existence is well established through theorems which assert them a topological origin [1]. Also, potentially anomalous contributions to a given theory can be pointed out by renormalization algebraic techniques [2]. There remains however their explicit evaluation as a separate task. In the scope of renormalizable models there are many ways to calculate anomalies: for example by regularizing the divergent amplitude with the constraints imposed by the renormalization prescription [3]. In a pioneer work Gernstein and Jackiw give an analytical evaluation of several Ward Identities [4]. In this work they resort to ambiguities inherent to performing shifts in linearly divergent integrals to explain the fact that it is not possible to simultaneously satisfy all three Ward identities related to the anomalous pion decay [4]. This argument has ever since been quoted (and used) in many Field Theory text books on the subject. The essential difference between their approach and the one in refs. [3] is that renormalization is not invoked as an essential ingredient for handling the divergent integrals. The physical meaning of the arbitrariness one has in choosing the internal momentum routing in loops is the translational invariance of free fermion propagators and therefore ascribing a fixed value to this arbitrary number ultimately means breaking one of the most fundamental symmetries of

Q.F.T. In fact arbitrary momentum routings play no role in the evaluation of Ward identities when consistent regularizations are used, such as Dimensional Regularization [5], proper-time [6], Pauli-Villars [7], etc., since shifts in the integration variable are allowed. Unfortunately given the mathematical complexity of extending the definition of the  $\gamma_5$  matrix to  $\omega$  dimensions in Dimensional Regularization, an equally transparent discussion of the axial-vector-vector anomaly is not as easy [8]. Recently a technique has been proposed to manipulate divergent integrals directly in four dimensions which makes use only **implicitly** of a regulator and has proven to yield equivalent results as Dimensional Regularization with the advantage of being naturally applicable to amplitudes involving pseudo quantities.

It is the purpose of the present contribution to calculate the axial-vector-vector amplitude in the context of such scheme. The technique has been proposed in ref. [9] and tested in several contexts [10]. We summarize the main steps of the prescription which we will follow here.

After evaluating Dirac traces wherever the case may be, one separates their divergent and finite contents. This separation is effected by means of implicitly assuming some regulating function from which two general properties are essential: a) it should be even in momentum. b) a connection limit should exist. The presence of such a function is indicated in the integrals by the symbol  $\Lambda$ , but there is no need to specify it further. After assuming the presence of such a function use is made of mathematical identities, on the level of the integrand, to isolate divergent from finite contributions, (which contain the physics of the amplitude). The manipulations used are similar in spirit to those employed in BPHZ [11], with the essential difference that no subtraction is performed. The purpose here is to isolate the divergent content of the Feynman diagrams to be independent of external momenta and establish the set of basic divergent integrals for every theory. In this context renormalization is achieved without resorting to specific regularizations, but simply upon identifying the set of basic divergent objects, and proceeding for its reparametrization. The regularizing function is only implicitly assumed.

Moreover, as extensively discussed in [10],[12] there is a set of three Consistency Relations (CR) which should be satisfied in order to avoid ambiguities of any kind. They are

$$\int_{\Lambda} \frac{d^4 k}{(2\pi)^4} \frac{k_{\mu} k_{\nu}}{(k^2 - m^2)^2} = \frac{g_{\mu\nu}}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)} = \frac{g_{\mu\nu}}{2} I_{quad}, \quad (1)$$

$$\int_{\Lambda} \frac{d^4 k}{(2\pi)^4} \frac{4k_{\alpha} k_{\beta}}{(k^2 - m^2)^3} = g_{\alpha\beta} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} = g_{\alpha\beta} I_{log}, \quad (2)$$

$$\int_{\Lambda} \frac{d^4 k}{(2\pi)^4} \frac{k_{\mu} k_{\nu} k_{\alpha} k_{\beta}}{(k^2 - m^2)^4} = \frac{g_{\alpha\beta}}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{k_{\mu} k_{\nu}}{(k^2 - m^2)^3}. \quad (3)$$

$$I_{quad}(m^2) = \int_{\Lambda} \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)}, \quad (4)$$

$$I_{log}(m^2) = \int_{\Lambda} \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2}, \quad (5)$$

These relations must be satisfied by any regularization prescription which aims at consistency. Of course, they are satisfied by Dimensional Regularization, since they represent the 4-D equivalent of respecting translational invariance in perturbative expansions,

mathematically materialized in the independence of arbitrary momentum routing in the internal loop lines. The existence of (at least) one such 4-D regulator has been proven in ref. [12] and therefore we do not discuss the validity of the Consistency Relations any longer. The obvious advantage of the present scheme is the fact that no special treatment needs to be given to the  $\gamma_5$  matrix.

In what follows we show how the anomalous pion decay comes about in 4-D in a scheme where surface terms related to a specific choice of the momentum routing are absent by construction. A consistent treatment is given to divergent integrals and renormalizability is only invoked after all calculations have been performed.

We need an explicit evaluation of the following amplitude

$$T_{\lambda\mu\nu}^{AVV} = e^2 \int \frac{d^4k}{(2\pi)^4} \text{Tr}\{\gamma^\lambda \gamma_5 (\not{k} + \not{p} - m)^{-1} \gamma^\mu (\not{k} - m)^{-1} \gamma^\nu (\not{k} - \not{q} - m)^{-1}\} \quad (6)$$

where the capital letters  $AVV$  stand for Axial-Vector-Vector respectively and  $\gamma_\mu$ , are the usual Dirac matrices and  $p, q$  the external momenta. In order to obtain an analytical expression for the amplitude in eq. (4) and verify the relative Ward Identities with all external momenta off the mass shell it is enough to evaluate the following three integrals.

$$I_1 = \int_{\wedge} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)[(p+k)^2 - m^2][(q+k)^2 - m^2]} \quad (7)$$

$$I_2 = \int_{\wedge} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{(k^2 - m^2)[(p+k)^2 - m^2][(q+k)^2 - m^2]} \quad (8)$$

and

$$I_3 = \int_{\wedge} \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - m^2)[(p+k)^2 - m^2][(q+k)^2 - m^2]} \quad (9)$$

The first two are finite and can be evaluated essentially by any method to yield

$$I_1 = \frac{i}{(4\pi)^2} \xi_{00}(p, q) \quad (10)$$

$$I_2 = -\frac{i}{(4\pi)^2} \{q_\mu \xi_{10}(p, q) + p_\mu \xi_{01}(p, q)\} \quad (11)$$

where the functions  $\xi_{nm}$  are defined in the Appendix.

The last integral eq. (7) is logarithmic divergent and here care must be exercised if one does not consider eventual subtractions from the beginning. We give the steps for the evaluation of (7) according to our prescription: we first use a mathematical identity at the level of the integrand in order to isolate the purely (external momenta independent) part of the integral,

and

$$I_3 = I_{\mu\nu}^{div} + I_{\mu\nu}^{fin} \quad (12)$$

where

$$I_{\mu\nu}^{div} = \frac{g_{\mu\nu}}{4} \int_{\wedge} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \quad (13)$$

$$\begin{aligned}
I_{\mu\nu}^{fin} = & - \frac{i}{(4\pi)^2} \{ g_{\mu\nu} [\frac{1}{2} Z_0((p-q)^2; m^2) \\
& - (\frac{1}{2} + m^2 \xi_{00}(p, q)) + \frac{q^2}{2} \xi_{10}(p, q) + \frac{p^2}{2} \xi_{01}(p, q)] \\
& - p_\mu p_\nu \xi_{02}(p, q) - q_\mu q_\nu \xi_{20}(p, q) \\
& - (p_\mu q_\nu - p_\nu q_\mu) \xi_{11}(p, q) \}
\end{aligned} \tag{14}$$

where the functions  $Z_k$  and  $\xi_{nm}$  depend on the external momenta  $p, q$  and on the mass  $m$ , see Appendix.

In order to verify the Ward identities we need to observe the following relations between the functions  $\xi_{nm}$ .

$$p^2 \xi_{10}(p, q) - p \cdot q \xi_{01}(p, q) = \frac{1}{2} \{ Z_0(q^2; m^2) - Z_0(p \cdot q; m^2) + p^2 \xi_{00}(p, q) \} \tag{15}$$

$$q^2 \xi_{11}(p, q) + p \cdot q \xi_{02}(p, q) = \frac{1}{2} \left\{ \frac{-Z_0(p-q)^2; m^2}{2} + \frac{Z_0(p^2; m^2)}{2} + q^2 \xi_{01}(p, q) \right\} \tag{16}$$

$$q^2 \xi_{20}(p, q) + p \cdot q \xi_{11}(p, q) = \frac{1}{2} \left\{ - \left[ \frac{1}{2} + m^2 \xi_{00}(p, q) \right] + \frac{p^2}{2} \xi_{01}(p, q) + \frac{3q^2}{2} \xi_{10}(p, q) \right\} \tag{17}$$

$$p^2 \xi_{02}(p, q) + p \cdot q \xi_{11}(p, q) = \frac{1}{2} \left\{ - \left[ \frac{1}{2} + m^2 \xi_{00}(p, q) \right] + \frac{q^2}{2} \xi_{10}(p, q) + \frac{3p^2}{2} \xi_{01}(p, q) \right\} \tag{18}$$

$$p^2 \xi_{11}(p, q) + p \cdot q \xi_{20}(p, q) = \frac{1}{2} \left\{ - \frac{1}{2} Z_0((p-q)^2; m^2) + \frac{1}{2} Z_0(q^2; m^2) + p^2 \xi_{10}(p, q) \right\} \tag{19}$$

After long and tedious but straightforward algebraic procedure we get

$$\begin{aligned}
T_{\lambda\mu\nu}^{AVV} &= e^2 \int \frac{d^4 k}{(2\pi)^4} Tr \{ \gamma^\lambda \gamma_5 (\not{k} + \not{p} - m)^{-1} \gamma^\mu (\not{k} - m)^{-1} \gamma^\nu (\not{k} - \not{q} - m)^{-1} \} \\
&= e^2 \{ \epsilon_{\lambda\mu\nu\omega} (p_\omega - q_\omega) F_1(p, q) \\
&+ p_\omega q_\phi \{ [\epsilon_{\lambda\nu\omega\phi} q^\mu + \epsilon_{\lambda\mu\omega\phi} q^\nu] F_2(p, q) \\
&+ [\epsilon_{\lambda\nu\omega\phi} p^\mu + \epsilon_{\lambda\mu\omega\phi} p^\nu] F_3(p, q) \\
&+ \epsilon_{\mu\nu\omega\phi} (p^\lambda F_4(p, q) + q^\lambda F_5(p, q)) \\
&+ \epsilon_{\lambda\mu\nu\omega} p_\omega F_6(p, q) - \epsilon_{\lambda\mu\nu\omega} q_\omega F_7(p, q) \}
\end{aligned} \tag{20}$$

where  $A$  stands for axial vector and

$$F_1(p, q) = -\frac{1}{4\pi^2} \left[ \frac{Z_0}{4} ((p+q)^2; m^2) - \frac{1}{4} + \frac{m^2 \xi_{00}(p, q)}{2} + \frac{q^2 \xi_{01}(p, q) + p^2 \xi_{10}(p, q)}{4} \right] \tag{21}$$

$$F_2(p, q) = \frac{1}{4\pi^2} [\xi_{01}(p, q) - \xi_{02}(p, q) - \xi_{11}(p, q)] \tag{22}$$

$$F_3(p, q) = -\frac{1}{4\pi^2} [\xi_{11}(p, q) + \xi_{20}(p, q) - \xi_{10}(p, q)] \tag{23}$$

$$F_4(p, q) = -\frac{1}{4\pi^2} [\xi_{11}(p, q) + \xi_{10}(p, q) - \xi_{20}(p, q)] \quad (24)$$

$$F_5(p, q) = -\frac{1}{4\pi^2} [\xi_{11}(p, q) + \xi_{01}(p, q) - \xi_{02}(p, q)] \quad (25)$$

$$F_6(p, q) = -\frac{1}{4\pi^2} \left[ -\frac{Z_0(p^2; m^2)}{4} - \frac{(p+q)^2}{2} \xi_{10}(p, q) \right] \quad (26)$$

$$F_7(p, q) = -\frac{1}{4\pi^2} \left[ \frac{Z_0(q^2; m^2)}{4} - \frac{(p+q)^2}{2} \xi_{01}(p, q) \right] \quad (27)$$

Now, in the context of a renormalizable theory such as the Linear Sigma Model there is one step still missing: the divergent contribution  $I_{log}$  must be renormalized away while a finite part must be fixed in a **unique** way [4]. At this point the essential meaning of the word anomaly appears very clearly: it is impossible to renormalize this amplitude simultaneously preserving all three Ward-Identities.

Note that in the present prescription the evaluation of eq. (0.20) which contains a bilinear in the numerator could not be performed in a different way, e.g., by considering contractions of the indices,  $\mu, \nu$  since here this procedure is not consistent with the CR eqs. (0.1-0.3). The difference between the two calculations is a constant and therefore it is of no importance in the context of renormalizable theories.

In summary we have calculated the triangle axial-vector-vector amplitude with all external momenta off shell which is analytic up to one scalar integral  $\xi_{00}$ , related to the Spence function. We have shown that there exist relations between the functions used to systematize the results which are essential to verify the Ward identities. Our calculation is performed within a 4-D scheme where an eventual regulator needs never be explicitated and odd numbers of  $\gamma_5$  matrices are easy to handle.

## Appendix

### General Integrals for the Finite Content of One Loop Amplitudes

#### The Functions $Z_k(p^2; m^2)$

We define

$$Z_k(p^2; m^2) = \int_0^1 dz z^k \ln \left( \frac{p^2 z(1-z) - m^2}{-m^2} \right) \quad (28)$$

where  $k$  is an integer,  $m$  is the mass parameter which appears in the propagators,  $p$  is some external momentum.

#### The Functions $\xi_{nm}$ :

When we consider one loop Feynman integrals associated to three point functions with two external momenta and three masses, the finite parts of the amplitudes are always related to the following general structures

$$\xi_{nm}(p, q) = \int_0^1 dz \int_0^{1-z} dy \frac{z^n y^m}{Q(y, z)} \quad (29)$$

where

$$Q = (y, z) = p^2 y(1 - y) + y + q^2 z(1 - z) + -m_1^2 - 2 p.q yz \quad (30)$$

In what follows we present the analytical expressions for equal masses and the  $k = 0, 1, 2$  cases, which will appear in our calculations. An extensive account of such structures in more general situations is given in reference [9]. All the  $\xi_{nm}$  functions we need will turn out to become linear combinations of the  $Z_k$  functions defined previously and the function  $\xi_{00}$ , related to the Spence function,

$$\frac{i}{(4\pi)^2} \xi_{00}(p, q) = \frac{Z_{-1}((p - q)^2; m^2)}{(p - q)^2} = \int_0^1 \frac{dz}{z} \ln\left(\frac{(p - q)^2 z(1 - z) - m^2}{-m^2}\right) \quad (31)$$

The procedure is the same as that of the preceeding section. We have

$$\begin{aligned} \xi_{01}(p, q) &= \left\{ \frac{p^2 q^2}{p^2 q^2 - (p.q)^2} \right\} \left\{ \frac{(p^2 - p.q)}{2p^2.q^2} [(-)Z_0(m^2, (p - q)^2; m^2)] \right. \\ &+ \frac{1}{2p^2} [Z_0(p^2; m^2)] \\ &+ \frac{(p.q)}{2p^2.q^2} [(-)Z_0(q^2; m^2)] \\ &+ \left. \frac{(p^2 - p.q)}{2p^2} [\xi_{00}(p, q)] \right\} \end{aligned} \quad (32)$$

$$\xi_{10}(p, q) = \xi_{01}(q, p) \quad (33)$$

$$\begin{aligned} \xi_{02}(p, q) &= \left\{ \frac{p^2 q^2}{p^2 q^2 - (p.q)^2} \right\} \left\{ \frac{(p.q)}{4p^2 q^2} [Z_0((p - q)^2; m^2) - [Z_0(p^2; m^2)]] \right. \\ &+ \frac{(p^2 - p.q)}{2p^2} [\xi_{01}(p, q)] - \frac{1}{2p^2} \left[ \frac{1}{2} + m^2 \xi_{00}(p, q) \right] \\ &+ \left. \frac{1}{4p^2} [q^2(\xi_{10}(p, q)) + p^2(\xi_{01}(p, q))] \right\} \end{aligned} \quad (34)$$

$$\xi_{20}(p, q) = \xi_{02}(q, p) \quad (35)$$

$$\begin{aligned} \xi_{11}(p, q) &= \left\{ \frac{p^2 q^2}{p^2 q^2 - (p.q)^2} \right\} \left\{ (-) \frac{1}{4q^2} [Z_0((p - q)^2; m^2) - Z_0(p^2; m^2)] \right. \\ &+ \frac{(q^2 - p.q)}{2q^2} [\xi_{01}(p, q)] + \frac{(p.q)}{2p^2.q^2} \left[ \frac{1}{2} + m^2 \xi_{00}(p, q) \right] \\ &+ \left. - \frac{(p.q)}{4p^2 q^2} [q^2(\xi_{10}(p, q)) + p^2(\xi_{01}(p, q))] \right\} \end{aligned} \quad (36)$$

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